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The existence of Bogomolny decomposition by means of strong necessary conditions

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Abstract
The concept of strong necessary conditions for the extremum of a functional to exist, has been applied to analyse the existence of the Bogomolny decomposition for a system of two coupled nonlinear partial differential equations in $1 + 1$ dimensions. A general condition for a derivativeless term has been derived for both hyperbolic and elliptic systems. In the case of independent equations the Bogomolny equations become Bäcklund transformations. Illustrative examples are presented for the particular form of derivativeless terms.

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1. Introduction

In the study of some nonlinear models solutions can be obtained by considering first-order differential equations (Bogomolny equations), instead of more complicated Euler–Lagrange equations [1–3]. The Bogomolny equations play a special role in the study of topological solitons. In recent years there have been numerous studies on the soliton solutions of Chern–Simons gauge theories, the Landau–Ginzburg model and the Maxwell–Chern–Simons theory [1, 4, 5]. The traditional method of deriving Bogomolny equations is based on transforming an expression for the energy of a field configuration into a positively determined form, for which the lower bound has a topological nature. A more powerful method than the traditional one is $N = 2$ supersymmetric extension of the investigated model [6–8]. Practising with the Bogomolny decomposition shows that this is a very effective tool for solving nonlinear partial differential equations of motion. The aim of this paper is to derive the conditions for existence of the Bogomolny decomposition by means of strong necessary conditions [9–12].

We consider systems of two coupled hyperbolic and two coupled elliptic nonlinear partial differential equations. The coupling term in both cases is derivativeless. Next, we assume that the system of equations considered corresponds to the Euler equations for extremals of a functional. We derive this functional and its variation. The necessary conditions for the extremum should not be formulated with the aid of the Euler equations that lead to the second-order partial differential equations. Instead of the Euler equations we apply the concept of strong necessary conditions, which leads to the first-order partial differential equations. This concept is based on the assumption that variations of all functional arguments and their derivatives are independent. The variational problem based on this concept guarantees that the set of solutions of new equations is included in the solution set resulting from the Euler equations. However, there are two consequences of this assumption.

(1) The resulting equations for the extremals are of first order. In the general case we obtain the partial differential equations of one order lower than the Euler equations.
(2) In the general case the resulting equations are trivial with respect to the set of solutions.

In order to obtain non-trivial equations of first order we apply to the functional the gauge transformation generated by the topological invariants. By definition this transformation does not change the Euler equation. When we apply the concept of independent variations to the gauge-transformed functional we derive the non-trivial equations which establish the Bogomolny decomposition.

The paper is organized as follows. In section 2 we briefly present the concept of strong necessary conditions for the problem of the extremum of a functional. Sections 3 and 4 contain considerations concerning the existence of the Bogomolny decomposition of the hyperbolic and elliptic systems. In section 4 we illustrate the results obtained with four examples. In the summary section we discuss the algorithmic structure of the derived formalism and point out the possibility of constructing a method which is a combination of both the Bogomolny decomposition and the Bäcklund transformations.

2. Concept of strong necessary conditions

The concept strong necessary conditions is based on the assumption that the equation considered results from the necessary conditions for an extremum of a functional to exist:

$$\delta\Phi[u] = 0 \tag{1}$$

where

$$\Phi[u] = \int_{E^2} F(u, u_{,x}, u_{,t})\,\mathrm{d}x\,\mathrm{d}t \tag{2}$$

and

$$u(\cdot, t) \in C^1 \qquad u(x, \cdot) \in C^1.$$

This condition and the assumption for the fixed boundaries lead to the Euler–Lagrange equation:

$$F_{,u} - D_x F_{,u_{,x}} - D_t F_{,u_{,t}} = 0. \tag{3}$$

However, equation (1) can be satisfied by assuming strong necessary conditions:

$$F_{,u_{,x}} = 0 \tag{4}$$

$$F_{,u_{,t}} = 0 \tag{5}$$

$$F_{,u} = 0. \tag{6}$$

Note that all of the solutions of equations (4)–(6) satisfy equation (3). However, in most cases the set of solutions of equations (4)–(6) is trivial (u = constant) or empty. In order to extend this set to a non-trivial one we use the gauge transformation of equation (2)

$$\Phi \to \Phi + I \tag{7}$$

and instead of equation (3) we apply equations (4)–(6). The scaling functional I is invariant with respect to the local variation of $u(x,t)$: $\delta I \equiv 0$. Therefore, the Euler–Lagrange equations resulting from the extremum of Φ and the extremum of $\Phi + I$ are equivalent. However, equations (4)–(6) are not invariant with respect to $\Phi \to \Phi + I$, i.e. the gauge transformation contributes to the strong necessary condition. This contribution can extend the subset of solutions to the non-trivial one. In this way, we derive simpler differential equations for extremals of the Φ solutions, which form a subset of the solutions of the Euler–Lagrange equation.

3. The existence of the Bogomolny decomposition for hyperbolic systems

Let us consider a hyperbolic system of two coupled nonlinear partial differential equations:

$$u_{,xt} = -\frac{\partial V(u,v)}{\partial u} \qquad v_{,xt} = \frac{\partial V(u,v)}{\partial v} \tag{8}$$

and formulate the following problem: what are the forms of $V(u, v)$ for which (8) satisfies the Bogomolny equations? The strong necessary conditions arises from the generating functional:

$$\Phi[u,v] = \int_{E^2} \left[\frac{1}{2}(u_{,x}u_{,t} - v_{,x}v_{,t}) - V(u,v)\right] \mathrm{d}x\,\mathrm{d}t. \tag{9}$$

For the purpose of the strong necessary conditions concept we generate the gauge transformation with

$$I_1 = \int_{E^2} G_1(u,v)(u_{,x}v_{,t} - u_{,t}v_{,x})\,\mathrm{d}x\,\mathrm{d}t \tag{10}$$

$$I_2 = \int_{E^2} D_x G_2(u,v)\,\mathrm{d}x\,\mathrm{d}t \tag{11}$$

$$I_3 = \int_{E^2} D_t G_3(u,v)\,\mathrm{d}x\,\mathrm{d}t. \tag{12}$$

Applying the concept of strong necessary conditions to the gauge-transformed Φ:

$$\Phi^* = \Phi + I_1 + I_2 + I_3 \tag{13}$$

we derive the following field equations:

$$u : -\frac{\partial V(u,v)}{\partial u} + G_{1,u}(u_{,x}v_{,t} - u_{,t}v_{,x}) + G_{2,uu}u_{,x} + G_{2,uv}v_{,x} + G_{3,uu}u_{,t} + G_{3,uv}v_{,t} = 0 \tag{14}$$

$$v : -\frac{\partial V(u,v)}{\partial v} + G_{1,v}(u_{,x}v_{,t} - u_{,t}v_{,x}) + G_{2,uv}u_{,x} + G_{2,vv}v_{,x} + G_{3,vu}u_{,t} + G_{3,vv}v_{,t} = 0 \tag{15}$$

$$u_{,x} : \tfrac{1}{2}u_{,t} + G_1 v_{,t} + G_{2,u} = 0 \tag{16}$$

$$v_{,x} : -\tfrac{1}{2}v_{,t} - G_1 u_{,t} + G_{2,v} = 0 \tag{17}$$

$$u_{,t} : \tfrac{1}{2}u_{,x} - G_1 v_{,x} + G_{3,u} = 0 \tag{18}$$

$$v_{,t} : -\tfrac{1}{2}v_{,x} + G_1 u_{,x} + G_{3,v} = 0. \tag{19}$$

Equations (14)–(19) must be self-consistent. Note that equations (16)–(19) characterize all hyperbolic (1 + 1)-dimensional partial differential equations (PDEs) of the type (8) because they do not depend on $V(u, v)$. Formally, we have six simultaneous equations for the five unknown functions: u, v, G_1, G_2, G_3. The system (14)–(19) becomes the Bogomolny equations if there exists as ansatz for G_1, G_2, G_3 for which the above equations reduce to two independent equations for u and v [9, 12]. Reduction of the number of independent equations by the appropriate choice of G_1, G_2, G_3 plays an essential role in this procedure. Only for very special $V(u, v)$ does such an ansatz exist. In most cases of $V(u, v)$ the system (14)–(19) cannot be reduced to the Bogomolny equations. However, even in this case one can use equations (14)–(19) to derive at least a particular set of solutions to equation (8).

We treat all of the unknown functions (u, v, G_1, G_2, G_3) as equivalent dependent variables governed by the set of equations (14)–(19). In order to make them equivalent we have to transform (14)–(19) into the equivalent system, which contains $D_x G_i$ and $D_t G_i$ instead of $G_{i,u}$ and $G_{i,v}$. We apply two transformations. The first one consists of integration of (14) with respect to u and (15) with respect to v:

$$-V(u, v) + G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_2 + D_t G_3 = \mathcal{D}(u_{,x}, u_{,t}, v_{,x}, v_{,t}) \tag{20}$$

where $\mathcal{D}$ is an arbitrary function. The remaining four equations (16)–(19) are transformed into the following set:

$$\tfrac{1}{2}(u_{,t}u_{,x} - v_{,t}v_{,x}) + G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_2 = 0 \tag{21}$$

$$\tfrac{1}{2}\left(u_{,t}^2 - v_{,t}^2\right) + D_t G_2 = 0 \tag{22}$$

$$\tfrac{1}{2}\left(u_{,x}^2 - v_{,x}^2\right) + D_x G_3 = 0 \tag{23}$$

$$\tfrac{1}{2}(u_{,t}u_{,x} - v_{,t}v_{,x}) + G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_t G_3 = 0. \tag{24}$$

Equation (21) has been obtained by adding (16) to (17) multiplied by $u_{,x}$ and $v_{,x}$ respectively. Equations (22)–(24) have been derived in an analogous way. Equations (20)–(24) establish a set of five equations for the five unknown functions. We call equations (20)–(24) the *divergence representation* of equations (14)–(19). It results from the self-consistency of equations (16)–(19) that all of the equations (20)–(24) must be self-consistent. We show in further considerations that equations (21)–(24) create the necessary conditions for integrability, whereas equation (20) corresponds to the sufficient condition. In order to satisfy equations (21)–(24) we have to reduce their left-hand sides to divergences. Thus, we factorize them using an appropriate choice of G_1:

$$\tfrac{1}{2}(u_{,t} - v_{,t})(u_{,x} + v_{,x}) = -D_x G_2 \tag{25}$$

$$\tfrac{1}{2}(u_{,t} - v_{,t})(u_{,t} + v_{,t}) = -D_t G_2 \tag{26}$$

$$\tfrac{1}{2}(u_{,x} - v_{,x})(u_{,x} + v_{,x}) = -D_x G_3 \tag{27}$$

$$\tfrac{1}{2}(u_{,t} - v_{,t})(u_{,x} + v_{,x}) = -D_t G_3 \tag{28}$$

where $G_1 = -\frac{1}{2}$. Now we can see that the left-hand sides of equations (25) and (26) are divergences if

$$\frac{u_{,t} - v_{,t}}{\sqrt{2}} = \mathcal{P}(v + u) \tag{29}$$

and analogously, the left-hand sides of equations (27) and (28) are divergences if:

$$\frac{u_{,x} + v_{,x}}{\sqrt{2}} = \mathcal{Q}(v - u) \tag{30}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are arbitrary functions. Equations (29) and (30) establish the necessary conditions for integrability of equations (20)–(24), whereas the sufficient condition will result from equation (20). Substituting equations (25) and (28) into equation (20), we obtain its simpler form:

$$V(u, v) = \mathcal{C}(u_{,x}, u_{,t}, v_{,x}, v_{,t}) \tag{31}$$

where $\mathcal{C}$ is an arbitrary function. Because of this freedom the solution set of (31) is, in general, wider than the solution set of (14) and (15). In order to guarantee the equivalence of (14), (15) and (20) in the sense of the solution sets, we reduce equation (20) to the Hamilton–Jacobi equation by an appropriate choice of $\mathcal{C}$:

$$V(u, v) = \tfrac{1}{2}(u_{,t} - v_{,t})(u_{,x} + v_{,x}) \tag{32}$$

where $\mathcal{C}$ has been determined from a comparison of (31) with the Hamilton–Jacobi equation for the system defined by the action functional Φ^* [13–15]:

$$\mathcal{H}(u, v, u_{,x}, v_{,x}, \pi_u, \pi_v) = 0 \tag{33}$$

where $\mathcal{H}$ is the Hamiltonian density:

$$\mathcal{H}(u, v, u_{,x}, v_{,x}, \pi_u, \pi_v) = \pi_u u_{,t} + \pi_v v_{,t} - \mathcal{L}. \tag{34}$$

The Lagrangian density is defined by (9) and (13):

$$\Phi^* = \int_{E^2} \mathcal{L}\, \mathrm{d}x\, \mathrm{d}t. \tag{35}$$

The canonical momenta are defined as follows: $\pi_u = \frac{\partial}{\partial u_{,t}}\mathcal{L}$ and $\pi_v = \frac{\partial}{\partial v_{,t}}\mathcal{L}$. Expanding equation (33) we obtain just (32). Substituting equations (29) and (30) into equation (32) we derive a sufficient integrability condition for equations (20)–(24):

$$V(u, v) = \mathcal{P}(u + v)\mathcal{Q}(v - u). \tag{36}$$

Note that equations (29) and (30) with $\mathcal{P}$ and $\mathcal{Q}$ subjected to equation (36) establish the Bogomolny equations for (8). *In the particular case where $V(u, v) = p(u) - q(v)$ equation (8) become independent*:

$$u_{,xt} = -\frac{\partial p(u)}{\partial u} \qquad v_{,xt} = -\frac{\partial q(v)}{\partial v} \tag{37}$$

and equations (29) and (30) with $\mathcal{P}$ and $\mathcal{Q}$ subjected to $p(u) - q(v) = \mathcal{P}(u + v)\mathcal{Q}(v - u)$ become the Bäcklund transformations for equation (37). To the best of our knowledge this is the first example so far that presents the relation between the Bogomolny equations and the Bäcklund transformations. The existence of such a relation has been suggested recently in [12].

Other integrability conditions can be derived for the following version of equation (8):

$$u_{,xt} = -\frac{\partial V(u, v)}{\partial v} \qquad v_{,xt} = -\frac{\partial V(u, v)}{\partial u}. \tag{38}$$

These equations are the Euler–Lagrange equations for the following functional of action:

$$\Phi^* = \int_{E^2} \left(\frac{1}{2}(u_{,x}v_{,t} + u_{,t}v_{,x}) - V(u, v)\right) \mathrm{d}x\, \mathrm{d}t + I_1 + I_2 + I_3. \tag{39}$$

The strong necessary conditions have the form:

$$-\frac{\partial V}{\partial u} + G_{1,u}(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_{2,u} + D_t G_{3,u} = 0 \tag{40}$$

$$-\frac{\partial V}{\partial v} + G_{1,v}(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_{2,v} + D_t G_{3,v} = 0 \tag{41}$$

$$\tfrac{1}{2}v_{,t} + G_1 v_{,t} + G_{2,u} = 0 \tag{42}$$

$$\tfrac{1}{2}u_{,t} - G_1 u_{,t} + G_{2,v} = 0 \tag{43}$$

$$\tfrac{1}{2}v_{,x} - G_1 v_{,x} + G_{3,u} = 0 \tag{44}$$

$$\tfrac{1}{2}u_{,x} + G_1 u_{,x} + G_{3,v} = 0. \tag{45}$$

Transforming equations (40)–(45) into the *divergence representation* we derive:

$$-V(u, v) + G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_2 + D_t G_3 = \mathcal{D}(u_{,x}, v_{,t}, u_{,t}, v_{,x}) \tag{46}$$

$$\tfrac{1}{2}(u_{,x}v_{,t} + u_{,t}v_{,x}) + G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_2 = 0 \tag{47}$$

$$u_{,t}v_{,t} + D_t G_2 = 0 \tag{48}$$

$$u_{,x}v_{,x} + D_x G_3 = 0 \tag{49}$$

$$\tfrac{1}{2}(u_{,x}v_{,t} + u_{,t}v_{,x}) + G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_t G_3 = 0. \tag{50}$$

We make equations (47)–(50) self-consistent by assuming that $G_1 = \frac{1}{2}$ and

$$v_{,t} = \mathcal{P}(u) \tag{51}$$

$$u_{,x} = \mathcal{Q}(v) \tag{52}$$

where $\mathcal{P}(u)$ and $\mathcal{Q}(v)$ are arbitrary functions that have to be used to make equation (46) an identity. In order to determine $\mathcal{D}$ in equation (46) we again use the Hamilton–Jacobi equation:

$$V(u, v) + u_{,x}v_{,t} = 0. \tag{53}$$

Substituting equations (51) and (52) into equation (53) we derive the condition for the existence of the Bogomolny decomposition:

$$V(u, v) + \mathcal{P}(u)\mathcal{Q}(v) = 0. \tag{54}$$

Equations (51) and (52) subjected to equation (54) represent the Bogomolny equations.

4. Bogomolny decomposition for elliptic PDEs in 1 + 1 dimensions

In this section we apply the strong necessary conditions to the analysis of the coupled pair of elliptic partial differential equations:

$$u_{,xx} + u_{,tt} = -\frac{\partial V(u, v)}{\partial u} \tag{55}$$

$$v_{,xx} + v_{,tt} = -\frac{\partial V(u, v)}{\partial v}. \tag{56}$$

In order to derive the Bogomolny equations from the strong necessary conditions concept we define the following gauge-transformed functional:

$$\Phi^*[u, v] = \int_{E^2} \left[\frac{1}{2}\left(u_{,x}^2 + u_{,t}^2\right) + \frac{1}{2}\left(v_{,x}^2 + v_{,t}^2\right) - V(u, v)\right] \mathrm{d}x\, \mathrm{d}t + I_1 + I_2 + I_3 \tag{57}$$

where I_1, I_2, I_3 are the topological invariants taken as in equations (10)–(12). Following the strong necessary condition concept we obtain the set of equations:

$$u : -\frac{\partial V(u, v)}{\partial u} + G_{1,u}(u_{,x}v_{,t} - u_{,t}v_{,x}) + (G_{2,uu}u_{,x} + G_{2,uv}v_{,x}) + (G_{3,uu}u_{,t} + G_{3,uv}v_{,t}) = 0 \tag{58}$$

$$v : -\frac{\partial V(u,v)}{\partial v} + G_{1,v}(u_{,x}v_{,t} - u_{,t}v_{,x}) + (G_{2,uv}u_{,x} + G_{2,vv}v_{,x}) + (G_{3,uv}u_{,t} + G_{3,vv}v_{,t}) = 0 \tag{59}$$

$$u_{,x} : u_{,x} + G_1 v_{,t} + G_{2,u} = 0 \tag{60}$$

$$v_{,x} : G_1 u_{,x} + v_{,t} + G_{3,v} = 0 \tag{61}$$

$$u_{,t} : u_{,t} - G_1 v_{,x} + G_{3,u} = 0 \tag{62}$$

$$v_{,t} : -G_1 u_{,t} + v_{,x} + G_{2,v} = 0. \tag{63}$$

Transforming equations (58)–(63) into the *divergence representation* we obtain:

$$-V(u,v) + G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_2 + D_t G_3 = \mathcal{D} \tag{64}$$

$$u_{,x}^2 + v_{,x}^2 + G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_2 = 0 \tag{65}$$

$$u_{,x}u_{,t} + v_{,x}v_{,t} + D_t G_2 = 0 \tag{66}$$

$$u_{,x}u_{,t} + v_{,x}v_{,t} + D_x G_3 = 0 \tag{67}$$

$$u_{,t}^2 + v_{,t}^2 + G_1(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_t G_3 = 0 \tag{68}$$

where $\mathcal{D}$ is an arbitrary function. In order to make them self-consistent we set $G_1 = 1$ and we obtain:

$$V(u,v) = \mathcal{C}(u_{,x}, u_{,t}, v_{,x}, v_{,t}) \tag{69}$$

$$u_{,x}(u_{,x} + v_{,t}) + v_{,x}(v_{,x} - u_{,t}) = -D_x G_2 \tag{70}$$

$$u_{,t}(u_{,x} + v_{,t}) + v_{,t}(v_{,x} - u_{,t}) = -D_t G_2 \tag{71}$$

$$u_{,x}(u_{,t} - v_{,x}) + v_{,x}(v_{,t} + u_{,x}) = -D_x G_3 \tag{72}$$

$$u_{,t}(u_{,t} - v_{,x}) + v_{,t}(v_{,t} + u_{,x}) = -D_t G_3. \tag{73}$$

Now we see that the left-hand sides of equations (70) and (71) are divergences if

$$u_{,x} + v_{,t} = -G_{2,u} \tag{74}$$

$$v_{,x} - u_{,t} = -G_{2,v}. \tag{75}$$

In order to satisfy equations (72) and (73) we have to assume

$$G_{2,u} = G_{3,v} \qquad G_{2,v} = -G_{3,u}. \tag{76}$$

Analogously, to the procedure described in section 2, we make equation (69) equivalent to equations (58) and (59) by reducing the solution set of equation (69) to the solution set of the Hamilton–Jacobi equation:

$$V(u,v) = \tfrac{1}{2}((u_{,x} + v_{,t})^2 + (u_{,t} - v_{,x})^2). \tag{77}$$

Substituting equations (74) and (75) into equation (77) we obtain the condition for the Bogomolny decomposition:

$$V(u,v) = \tfrac{1}{2}\left(G_{2,u}^2 + G_{2,v}^2\right) \qquad G_{2,uu} + G_{2,vv} = 0. \tag{78}$$

Equations (74)–(76) establish the Bogomolny equations. *In the particular case where* $V(u,v) = p(u) - q(v)$ *equations (55) and (56) become independent and equations (74) and (75) with* G_2 *subjected to equation (78) become the Bäcklund transformations*.

Cross-coupling in equations (55) and (56) leads to the following set of equations:

$$u_{,xx} + u_{,tt} = -\frac{\partial V(u,v)}{\partial v} \tag{79}$$

$$v_{,xx} + v_{,tt} = -\frac{\partial V(u,v)}{\partial u}. \tag{80}$$

Applying the investigation procedure derived in the previous cases we derive the *divergence representations* of the strong necessary conditions for equations (79) and (80):

$$V(u,v) = \mathcal{C}(u_{,x}, u_{,t}, v_{,x}, v_{,t}) \tag{81}$$

$$u_{,x}(v_{,x} + G_1 v_{,t}) + v_{,x}(u_{,x} - G_1 u_{,t}) = -D_x G_2 \tag{82}$$

$$u_{,t}(v_{,x} + G_1 v_{,t}) + v_{,t}(u_{,x} - G_1 u_{,t}) = -D_t G_2 \tag{83}$$

$$u_{,x}(v_{,t} + \lambda v_{,x}) + v_{,x}(u_{,t} - \lambda u_{,x}) = -D_x G_3 \tag{84}$$

$$u_{,t}(v_{,t} - G_1 v_{,x}) + v_{,t}(u_{,t} + G_1 u_{,x}) = -D_t G_3 \tag{85}$$

where $\mathcal{C}$ is an arbitrary function and λ is an arbitrary constant. It is impossible to make equations (82)–(85) self-consistent with two relations such as (74) and (75) expressed by two arbitrary functions of one variable, however, it can be done in two ways:

(1) with four relations involving four arbitrary functions of one variable;
(2) with four relations involving two functions of two variables.

Both ways lead to an over determined system of equations with a very particular set of solutions. Such a situation means that there is *no Bogomolny decomposition*.

5. Examples

Now we give four examples illustrating the above considerations for particular forms of potential $V(u,v)$, for which Bogomolny decomposition does exist.

5.1. Hyperbolic system

The first concerns the hyperbolic system corresponding to (38):

$$u_{,xt} = -\lambda \sin(u)\sinh(v) \qquad v_{,xt} = -\lambda \cos(u)\cosh(v). \tag{86}$$

Applying equations (54) and (51), (52) we derive the Bogomolny equations:

$$v_{,x} + \sqrt{\lambda}\sin(u) = 0 \qquad u_{,t} - \sqrt{\lambda}\cosh(v) = 0. \tag{87}$$

5.2. Elliptic system

We construct an elliptic example that is integrable using the Bogomolny equations starting from (78). Assuming $G_2(u,v) = \frac{1}{2}(f(u+\mathrm{i}v) + f(u-\mathrm{i}v))$ we derive the potential:

$$V(u,v) = \tfrac{1}{2} f'(z) f'(z^*) \tag{88}$$

where $z = u + \mathrm{i}v$. Substituting equation (88) into equations (55) and (56) we obtain the elliptic system:

$$u_{,xx} + u_{,tt} = -\tfrac{1}{2}[f'(z)f''(z^*) + f'(z^*)f''(z)]$$

$$v_{,xx} + v_{,tt} = -\frac{i}{2}[f'(z)f''(z^*) - f'(z^*)f''(z)]$$

which is integrable using the first-order differential equations:

$$u_{,x} + v_{,t} = -\tfrac{1}{4}(f'(z) + f'(z^*)) \qquad v_{,x} - u_{,t} = -\frac{i}{4}(f'(z) - f'(z^*)).$$

5.3. Intersecting of domain walls

These results can be directly applied to considerations of static intersecting domain-wall solutions for some field theory models [16–18]. In order to make direct use of the obtained results we extend the action functional (57) of the (1 + 1)-dimensional dynamical model to the static energy of the (2 + 1)-dimensional field theory:

$$\Phi^*[u,v] = \int_{E^2} \left[\frac{1}{2}\left(u_{,x}^2 + u_{,y}^2\right) + \frac{1}{2}\left(v_{,x}^2 + v_{,y}^2\right) + V(u,v) \right] \mathrm{d}x\,\mathrm{d}y + I_1 + I_2 + I_3. \tag{89}$$

Let us consider a polynomial potential possessing a triple degenerate minimum $V = 0$:

$$V = G_{2,u}^2 + G_{2,v}^2 \qquad G_{2,uu} + G_{2,vv} = 0 \tag{90}$$

where $G_2 = v - u^3 v + uv^3$. Combining dependent variables into the complex field $\psi = u + \mathrm{i}v$ we obtain the following form for V:

$$V = \psi^{*3}\psi^3 - \psi^3 - \psi^{*3} + 1. \tag{91}$$

Equation (91) possesses the following set of critical points: $\psi = 1, \omega, \omega^2$, which is isomorphic to the $\mathbf{Z}_3$ group of symmetry. There are therefore three possible domains and three types of domain wall separating them [16, 17]. Combining x and y into the complex variable $z = x + \mathrm{i}y$ we transform the Bogomolny equations (74) and (75) into the following forms:

$$2\psi_{,z} = \mathrm{i}(\psi^{*3} - 1) \tag{92}$$

$$2\psi^*_{,z^*} = -\mathrm{i}(\psi^3 - 1). \tag{93}$$

Equations (92) and (93) are invariant under the $\mathbf{Z}_3$ action: $(z, \psi) \to (\omega z, \omega\psi)$, so we are led to seek a $\mathbf{Z}_3$ invariant solutions such that $\psi \to 1$ as one goes to infinity inside the sector $-\frac{\pi}{6} < \arg z < \frac{\pi}{6}$, subject to the condition that $\arg z = \arg \psi$ on the boundary. By symmetry ψ must vanish at the origin and so $\psi \approx -\mathrm{i}z$ for small z [17].

This qualitative analysis shows that a stable static triple intersection does indeed exist. Therefore, metastable networks of domain walls should also exist [18].

5.4. Field equations associated with $\pi_3(S^2)$ homotopy group

A less trivial problem corresponds to a mapping of the three-dimensional space of independent variables into a two-dimensional sphere. Let us assume that all possible values of a continuous field establish a manifold isomorphic to S^2. This assumption is equivalent to the assumption of constant boundary conditions at infinity. Therefore, any continuous field function satisfying the boundary conditions can be classified by the homotopy class. The set of all of these classes (and the rules of superposition) establish the homotopy group $\pi_3(S^2)$. There are several important field models classified by $\pi_3(S^2)$: the static three-dimensional classical Heisenberg model and field models in (2 + 1)-dimensional space generating soliton equations. In this subsection we present some results for the Heisenberg model (σ-model) [19]. The model is governed by the following differential equation:

$$\triangle w - \frac{2w^*(\nabla w)^2}{1 + ww^*} = 0 \tag{94}$$

where $w = \frac{S^x + \mathrm{i}S^y}{1 + S^z}$, the S^x, S^y, S^z are components of the classical Heisenberg spin, normalized to a constant value: $(S^x)^2 + (S^y)^2 + (S^z)^2 = \text{constant}$. Equation (94) results from the least 'action' principle, where the 'action' is represented by the integral of the static energy:

$$H = \int_{E^3} \frac{\nabla w \nabla w^*}{(1 + ww^*)^2}\,\mathrm{d}^3x \tag{95}$$

where w is a complex field on E^3. It is sufficient to attach only the following invariants:

$$I_i = \int_{E^3} G_i(w, w^*)\epsilon^{ijk} w_{,x_j} w^*_{,x_k}\, \mathrm{d}^3x \tag{96}$$

where[4] $i, j, k = 1, 2, 3$,

$$I_{l+3} = \int_{E^3} D_{x_l} G_{l+3}(w, w^*)\, \mathrm{d}^3x \tag{97}$$

where $l = 1, 2, 3$. Complete list of the topological invariants should also contain the Hopf invariant [20]. However, for simplicity we assume in advance that its weight function is equal to zero. We derive the following strong necessary conditions:

$$-\frac{2w^*\nabla w\nabla w^*}{(1+ww^*)^3} + \epsilon^{ijk} G_{i,w}(w, w^*) w_{,x_j} w^*_{,x_k} + \sum_l D_{x_l} G_{l+3,w}(w, w^*) = 0 \tag{98}$$

$$\frac{w^*_{,x_p}}{(1+ww^*)^2} + \epsilon^{ipk} G_i(w, w^*) w^*_{,x_k} + G_{p+3,w}(w, w^*) = 0 \qquad \text{c.c.} \tag{99}$$

where $p = 1, 2, 3$. Equations (98) and (99) and the complex-conjugated (c.c.) equations must be self-consistent. This requirement determines $G_k(w, w^*)$:

$$G_k(w, w^*) = \frac{\mathrm{i}\lambda_k}{(1+ww^*)^2} \tag{100}$$

for $k = 1, 2, 3$ and $G_k(w, w^*) = 0$ for $k = 4, 5, 6$. Substituting equation (100) into equation (99) we obtain:

$$w^*_{,x_p} + \mathrm{i}\epsilon^{jpk}\lambda_j w^*_{,x_k} = 0 \qquad \text{c.c.} \tag{101}$$

The necessary condition for the non-trivial solutions of equation (99) gives the following constraint for λ_j:

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1. \tag{102}$$

The necessary condition (98) is automatically satisfied by (100)–(102). Therefore, equation (101) constrained to (102) establishes the Bogomolny equations. Solving equation (101) we derive:

$$w^* = w^*\left((\mathrm{i}\lambda_2 + \lambda_3\lambda_1)x_1 + (\mathrm{i}\lambda_1 - \lambda_3\lambda_2)x_2 + \left(\lambda_3^2 - 1\right)x_3\right) \qquad \text{c.c.} \tag{103}$$

6. Summary

In the case of coupled pairs of partial differential equations for two unknown functions in two-dimensional space and resulting from the least action principle, the investigations of integrability using the Bogomolny equations can be set within the framework of the following algorithm:

(a) construct the action functional (9);
(b) construct the topological invariants (10)–(12);
(c) perform the gauge transformation (13);
(d) derive the strong necessary conditions for the extremum of Φ^* equations (14)–(19);
(e) transform equations (14)–(19) to the *divergence representation* equations (20)–(24);
(f) set the necessary conditions for equations (21)–(24) to be divergences;
(g) derive the Hamilton–Jacobi equation (33);

[4] The summation convention used in this subsection is the following: equal indices are contracted only if they appear on the right-hand side of ϵ^{ijk}.

(h) comparing equation (31) with (32) to determine the free function $\mathcal{C}$;

(i) using equations (29) and (30) eliminate all partial derivatives from equation (32).

The relation between $V(u, v)$ and the arbitrary functions $\mathcal{P}$ and $\mathcal{Q}$ establishes a condition for the existence of the Bogomolny equations. This algorithm can be easily extended into systems of m partial differential equations for p unknown functions in an arbitrary n-dimensional space of independent variables. The simplest case occurs if $p = n$. Then we set up boundary conditions for which any continuous solution $u_1, u_2, \ldots, u_n$ generates a mapping belonging to the $\pi_n(S^n)$ homotopy group; therefore, we have to take into account all possible topological invariants [12]. Therefore, point (c) has to be extended:
(c′) construct the set of topological invariants:

$$
\begin{aligned}
&\binom{n}{1}^1 \text{invariants of the type} \int_{E^n} D_{x_i} G_i(u_1 \cdots u_n)\, \mathrm{d}^n x \\
&\binom{n}{2}^2 \text{invariants of the type} \int_{E^n} \epsilon^{a_1 a_2} u_{1,a_1} u_{2,a_2}\, \mathrm{d}^n x \\
&\binom{n}{3}^2 \text{invariants of the type} \int_{E^n} \epsilon^{a_1 a_2 a_2} u_{1,a_1} u_{2,a_2} u_{3,a_3}\, \mathrm{d}^n x \\
&\vdots \\
&\binom{n}{n}^2 \text{invariants of the type} \int_{E^n} \epsilon^{a_1 a_2 \cdots a_n} u_{1,a_1} u_{2,a_2} \cdots u_{n,a_n}\, \mathrm{d}^n x
\end{aligned}
\tag{104}
$$

where $\epsilon^{a_1 a_2 \cdots a_n}$ is a complete antisymmetric tensor. If necessary, one must supply equation (104) with the higher-order invariants with respect to the orders of derivatives in the dependent variables.

In the case of $p < n$ we have to also modify point (a):
(a′) supply the investigated system with some independent systems in such a way that the supplemented system will generate the $\pi_n(S^n)$ or $\pi_{2n-1}(S^n)$ homotopy group. This procedure is a combination of both the Bäcklund transformation and the Bogomolny decomposition. Note that the results of this paper show for the first time the transition between these two methods.

Finally, we make some remarks concerning the relation between the results described and the classes of functionals admitting a Bogomolny decomposition obtained from supersymmetric arguments [21, 22]. Let us point out where the relevant attributes of the supersymmetric theory (topological stability, the Bogomolny bound and the supermultiplet of the Bogomolny equations) are given in the method described. Topological stability is achieved (if necessary) by supplying the investigated system with some independent systems (see point (a′) of this section and [9, 12]). This extension enables one to construct the topological invariants which play the role of topological charges. In order to explain the Bogomolny relationships we return to section 4. The first two *strong necessary conditions* (58) and (59) result from the variations of equation (57) with respect to u and v. By integration they are transformed into the Hamilton–Jacobi equation (33) which gives the form of equation (77). The next four *strong necessary conditions* (60)–(63) play the role of the supermultiplet of the Bogomolny equations. Using equations (60)–(63) we eliminate from equation (77) all partial derivatives of the field variables u and v. Owing to the topological terms included in Φ^* and $\mathcal{H}$, the obtained equation is equivalent to the saturated Bogomolny bound (78). This is expressed by G_2 which plays the role of the superpotential. In summary, we see that the described method possesses all the attributes of the supersymmetric arguments leading to the

Bogomolny relationship. Therefore, the Bogomolny equations resulting from the concept of strong necessary conditions and the supersymmetric arguments must be equivalent.

References

[1] Bogomolny E B 1976 *Sov. J.–Nucl. Phys.* **24** 861
[2] Belavin A A, Polyakov A M, Schwartz A S and Tyupkin Yu S 1975 *Phys. Lett.* B **59** 85
[3] de Vega H J and Schaposnik F A 1976 *Phys. Rev.* D **14** 1100
[4] Hong J, Kim Y and Pac P Y 1990 *Phys. Rev. Lett.* **64** 2230
[5] Jackiw R and Weinberg E J 1990 *Phys. Rev. Lett.* **64** 2234
[6] Olive D and Witten E 1978 *Phys. Lett.* B **78** 97
[7] West P 1986 *Introduction to Supersymmetry and Supergravity* (Singapore: World Scientific)
[8] Damski B 2000 *Acta Phys. Polon.* B **31** 637
[9] Sokalski K, Wietecha T and Lisowski Z 2001 *Acta Phys. Polon.* B **32** 17
[10] Sokalski K 1979 *Acta Phys. Polon.* A **56** 571
[11] Sokalski K 1981 *Phys. Lett.* A **81** 102
[12] Sokalski K, Wietecha T and Lisowski Z 2001 *Acta Phys. Polon.* B **32** 2771
[13] Rund H 1966 *The Hamilton–Jacobi Theory in the Calculus of Variations* (London: Van Nostrand)
[14] Lévy P 1951 Problèmes Concrets d'Analyse Fonctionnelle *Les Espaces Abstraits et leur Theorie Consideree comme Introduction a L'analyse Generale* 2nd edn ed M Frechet (Paris: Gaulthier-Villars)
Pellegrino F 1951 Uncomplement sur les functionnelles analytiques *Les Espaces Abstraits et leur Theorie Consideree Comme Introduction a l'analyse Generale* 2nd edn ed M Frechet (Paris: Gaulthier-Villars)
[15] Lisowski Z and Sokalski K 1999 *J. Phys. A: Math. Gen.* **32** 5907
[16] Abraham E R C and Townsend P K 1991 *Nucl. Phys.* B **351** 313
[17] Gibbons G W and Townsend P K 1999 *Phys. Rev. Lett.* **83** 1727
[18] Carroll S M, Hellerman S and Trodden M 1999 *Preprint* hep-th/9905217
[19] Sokalski K 1984 *Acta Phys. Polon.* A **65** 457
[20] Flanders H 1963 *Differential Forms with Applications to the Physical Sciences* (New York: Academic) ch 6.3
[21] Hlousek Z and Spector D 1992 *Nucl. Phys.* B **370** 143
[22] Hlousek Z and Spector D 1993 *Nucl. Phys.* B **397** 173